

# Of Higgs, Unitarity and other Questions <sup>1</sup>

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**Abstract:** On the verge of conclusive checks on the Standard Model by the LHC, we discuss some of the basic assumptions. The reason for this analysis stems from a recent proposal of an Electroweak Model based on a nonlinearly realized gauge group  $SU(2) \otimes U(1)$ , where, in the perturbative approximation, there is no Higgs boson. The model enjoys the Slavnov-Taylor identities and therefore the perturbative unitarity. On the other hand, it is commonly believed that the existence of the Higgs boson is entangled with the property of unitarity, when high energy processes are considered. The argument is based mostly on the Froissart bound and on the *Equivalence Theorem*. In this talk we briefly review some of our objections on the validity of such arguments. Some open questions are pointed out, in particular on the limit of zero mass for the vector mesons and on the fate of the longitudinal polarizations.

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# 1 Introduction

The main assumptions for the construction of a massive Yang-Mills (YM) *local* quantum field theory are

1. Renormalizability
2. Unitarity
3. Spontaneous Breakdown of Symmetry.

The mass is derived from the interaction with the Higgs field

$$S_{SSB} = S_{YM} + \frac{\Lambda^{D-4}}{g^2} \int d^D x \frac{1}{4} \text{Tr} \left\{ \left| \partial_\mu \Omega - i A_\mu \Omega \right|^2 \right\} + S_{BS}, \quad (1)$$

where  $S_{BS}$  is the pure boson part of the action responsible for the nonzero vacuum expectation value of the Higgs boson field. For  $SU(2)$  the matrix  $\Omega$  may be parametrized by the real fields

$$\Omega = \phi_0 + i\tau_i \phi_i, \quad \phi_0 = h + 2v, \quad \langle h \rangle = 0, \quad M = gv \quad (2)$$

In this talk we focus mainly on the issue of unitarity and on its connection with the presence of a physical Higgs boson in the perturbative spectrum. In part one we consider a brief statement of the problem on general grounds, i.e. on the perturbative unitarity and its relationship with the optical theorem. In part two we derive relations between the amplitudes involving the scalar part of the vector mesons on the one hand and the Goldstone bosons on the other. These relations are somehow related to the so-called Equivalence Theorem in gauge theories. In part three we flash some of the work we did on the nonlinear sigma model and on the massive Yang-Mills theory in order to put on a subtraction procedure for these nonrenormalized theories.

## 2 Part One: Unitarity

The attention has been focused on the  $W_L W_L$  elastic scattering process for different reasons. At high energy ( $s, t \gg M_W^2$ ) some anomalous behavior is expected for the longitudinal polarization. The idea is to entangle the presence of the Higgs boson with the requirement of unitarity. The calculations often make use of the so called *Equivalence Theorem* [1]-[5].

### 2.1 Unitarity:

It is worth stressing the conceptual difference between the *Optical Theorem* for the  $S$ -matrix

$$S = \mathbb{I} - iT, \quad S^\dagger S = \mathbb{I}, \quad \implies \text{Im} T_{ii} \sim \sigma_{iT} \quad (3)$$

and *Perturbative Unitarity*

$$\sum_{j=0}^k S^{(j)\dagger} S^{(k-j)} = 0, \quad \forall k > 0, \quad (4)$$

where

$$S = \sum_{k=0}^{\infty} S^{(k)}, \quad S^{(0)} = \mathbb{I}. \quad (5)$$

For any finite order calculation  $S_{in} = \sum_{j=0}^k S_{in}^{(j)}$

$$\begin{aligned} \sum_n \left| \sum_{j=0}^k S_{in}^{(j)} \right|^2 &= \sum_n \sum_{l=0}^k \sum_{j=0}^k S_{in}^{(j)*} S_{in}^{(l-j)} + \sum_n \sum_{l=k+1}^{2k} \sum_{j=0}^k S_{in}^{(j+l)*} S_{in}^{(l-j)} \\ &= 1 + \sum_n \sum_{l=k+1}^{2k} \sum_{j=l-k}^k S_{in}^{(j)*} S_{in}^{(l-j)} \end{aligned} \quad (6)$$

There is always a violation of the Optical Theorem of order  $\mathcal{O}(k+1)$ .

The Optical Theorem has a meaning only if an operative definition of *forward* scattering exists. If long range interactions are present, then the forward amplitude is an elusive object. Only eq. (4) has a meaning.

### 3 Part Two: Equivalence Theorem

This part is devoted to the discussion of some aspects of the massive YM theory in the linear representation of the gauge group of local transformations (Higgs mechanism). Most of the results are also valid for the case in which the representation is nonlinear (Stückelberg mass).

#### 3.1 BRST Transformations:

The BRST differential  $\mathfrak{s}$  is obtained in the usual way by promoting the gauge parameters to the ghost fields  $c_a$  and by introducing the antighosts  $\bar{c}_a$  coupled in a BRST doublet to the Nakanishi-Lautrup fields  $b_a$ :

$$\begin{aligned} \mathfrak{s}\phi_a &= \frac{1}{2}\phi_0 c_a + \frac{1}{2}\epsilon_{abc}\phi_b c_c, & \mathfrak{s}\phi_0 &= -\frac{1}{2}\phi_a c_a \\ \mathfrak{s}A_{a\mu} &= (D_\mu[A]c)_a, & \mathfrak{s}\bar{c}_a &= b_a, & \mathfrak{s}b_a &= 0. \end{aligned} \quad (7)$$

In the above equation  $D_\mu[A]$  denotes the covariant derivative w.r.t.  $A_{a\mu}$ :

$$(D_\mu[A])_{ac} = \delta_{ac}\partial_\mu + \epsilon_{abc}A_{b\mu}. \quad (8)$$

The BRST transformation of  $c_a$  then follows by nilpotency,

$$\mathfrak{s}c_a = -\frac{1}{2}\epsilon_{abc}c_b c_c. \quad (9)$$

The tree-level vertex functional is

$$\begin{aligned}
\Gamma^{(0)} &= S_{SSB} + \frac{\Lambda^{(D-4)}}{g^2} \mathfrak{s} \int d^D x (\bar{c}_a \partial A_a) \\
&\quad + \frac{\Lambda^{(D-4)}}{g^2} \int d^D x (A_{a\mu}^* \mathfrak{s} A_a^\mu + \phi_a^* \mathfrak{s} \phi_a + \phi_0^* \mathfrak{s} \phi_0 + c_a^* \mathfrak{s} c_a) \\
&= S_{YM} + \frac{\Lambda^{(D-4)}}{g^2} \int d^D x \left( b_a \partial A_a - \bar{c}_a \partial_\mu (D^\mu [A] c)_a \right) \\
&\quad + \frac{\Lambda^{(D-4)}}{g^2} \int d^D x (A_{a\mu}^* \mathfrak{s} A_a^\mu + \phi_0^* \mathfrak{s} \phi_0 + \phi_a^* \mathfrak{s} \phi_a + c_a^* \mathfrak{s} c_a). \tag{10}
\end{aligned}$$

In  $\Gamma^{(0)}$  we have also included the antifields  $A_{a\mu}^*, \phi_0^*, \phi_a^*$  and  $c_a^*$  coupled to the nonlinear BRST variations of the quantized fields.

### 3.2 Slavnov-Taylor Identity (STI):

To simplify notations, we perform the substitution  $b_a \rightarrow \frac{g^2}{\Lambda^{(D-4)}} b_a$ . The STI are for the 1-PI functional (ZJ renormalization of composite operators) is

$$\int d^D x \left( \Gamma_{A_{a\mu}^*} \Gamma_{A_a^\mu} + \Gamma_{\phi_a^*} \Gamma_{\phi_a} + \Gamma_{\phi_0^*} \Gamma_{\phi_0} + \Gamma_{c_a^*} \Gamma_{c_a} + b_a \Gamma_{\bar{c}_a} \right) = 0, \tag{11}$$

where we use the notation

$$\Gamma_X \equiv \frac{\delta \Gamma}{\delta X}, \tag{12}$$

while for the generating functional of the connected amplitudes one has

$$\int d^D x \left( -W_{A_{a\mu}^*} J_{a\mu} - W_{\phi_a^*} K_a - W_{\phi_0^*} K_0 + W_{c_a^*} \bar{\eta}_a - W_{b_a} \eta_a \right) = 0 \tag{13}$$

We use the notations

$$W_{A_{a\mu}^* \dots} \equiv \frac{\delta^n W}{\delta A_{a\mu}^* \dots} = i^{n-1} \langle 0 | T((D^\mu [A] c)_a \dots) | 0 \rangle_C \tag{14}$$

for composite fields, while for elementary fields

$$W_{\underbrace{b_a \dots}_n} \equiv i^{n-1} \langle 0 | T(b_a \dots) | 0 \rangle_C. \tag{15}$$

The external field sources are

$$\int d^D x \left( J_{a\mu} A_a^\mu + K_a \phi_a + K_0 \phi_0 + \bar{c}_a \eta_a + \bar{\eta}_a c_a + J_{b_a} b_a \right). \tag{16}$$

### Landau Gauge Equation

The equation associated to the gauge fixing gives

$$\Gamma_{b_a} = \partial_\nu A_a^\nu \tag{17}$$

$$-J_{b_a} = \partial_\mu W_{A_a^\mu}. \quad (18)$$

The antighost equations can be derived from eqs. (11), (13), (17) and (18):

$$\Gamma_{\bar{c}_a} = \partial_\nu \Gamma_{A^* a \nu} \quad (19)$$

$$\eta = \partial^\nu W_{A_a^* \nu}. \quad (20)$$

From eqs. (13) and (20) one gets

$$\begin{aligned} W_{\phi^* \bar{c}} &= W_{\phi b} \\ W_{A_\mu^* \bar{c}} &= W_{A^\mu b} = -i \frac{p^\mu}{p^2}. \end{aligned} \quad (21)$$

### Some Basic Results

By a straightforward use of the above equations and of

$$\Gamma W = -\mathbb{I}, \quad (22)$$

one gets

$$W_{\phi b} = i \frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi \phi}} \frac{1}{p^2} \quad (23)$$

$$(p^\nu \Gamma_{A^\nu \phi})^2 + p^2 \Gamma_L \Gamma_{\phi \phi} = 0 \quad (24)$$

$$W_{A^\mu \phi} = 0, \quad W_L = 0, \quad W_{\phi \phi} = -\frac{1}{\Gamma_{\phi \phi}}. \quad (25)$$

### Free Fields

The 2-point 1-PI functions are given by

$$\begin{aligned} \Gamma_{A^\nu \phi} &= iM p_\nu, \quad \Gamma_{bb} = 0, \quad \Gamma_{A^\nu b} = i p_\nu, \\ \Gamma_{\phi \phi} &= p^2, \quad \Gamma_{\phi b} = 0, \quad \Gamma_L = M^2. \end{aligned} \quad (26)$$

Then

$$W_{A^\mu \phi} = 0 \quad (27)$$

and

$$\begin{aligned} W_{A^\mu b} &= -i \frac{p^\mu}{p^2}, \quad W_{\phi b} = \frac{M}{p^2} \\ W_L &= 0, \quad W_{\phi \phi} = -\frac{1}{p^2}. \end{aligned} \quad (28)$$

**Theorem** For  $m \geq 1$

$$\begin{aligned}
W_{b_{x_1} \dots b_{x_m}} &= 0, \\
W_{b_{x_1} \dots b_{x_m} \phi_{z_1}^* \bar{c}_{y_1} \dots \phi_{z_k}^* \bar{c}_{y_k}} &= 0, \\
W_{b_{x_1} b_{x_1} \dots b_{x_m} \phi_{w_1} \dots \phi_{w_n}} &= \sum_{i=1}^n W_{\phi_{w_i}^* \bar{c}_{y_1} \dots b_m \phi_{w_1} \dots \overset{\vee}{\phi_{w_i}} \dots \phi_{w_n}}, \\
\sum_{j=1}^k (-1)^j W_{b_{y_j} b_{x_1} \dots b_{x_m} \phi_{z_1}^* \dots \phi_{z_{k-1}}^* \bar{c}_{y_1} \dots \overset{\vee}{\bar{c}_{y_j}} \dots \bar{c}_{y_k} \phi_{w_1} \dots \phi_{w_n}} \\
+ \sum_{i=1}^n W_{b_{x_1} \dots b_{x_m} \phi_{z_1}^* \dots \phi_{z_{k-1}}^* \phi_{w_i}^* \bar{c}_{y_1} \dots \bar{c}_{y_k} \phi_{w_1} \dots \overset{\vee}{\phi_{w_i}} \dots \phi_{w_n}} &= 0
\end{aligned} \tag{29}$$

where  $\vee$  marks omitted symbols. Proof: just use the STI.

Eq. (29) is easily generalized to the case where any number of external physical legs are added (via the reduction formulae formalism).

### 3.3 $b$ -insertions

The quantity

$$R \equiv i \frac{p^\nu \Gamma_{\phi A^\nu}}{M \Gamma_{\phi\phi}} \Big|_{p^2=0} = \frac{p^2}{M} W_{b\phi} \Big|_{p^2=0} \tag{30}$$

will appear all over again (at the tree level  $R = 1$ ). The pole contribution gives

$$\begin{aligned}
\lim_{p^2=0} p^2 W_{b(p) \dots} &= \left( i \frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} W_{\widehat{\phi(p) \dots}} + i p^\mu W_{\widehat{A^\mu(p) \dots}} \right) \Big|_{p^2=0} \\
&= \left( -M R \Gamma_{\phi\phi} W_{\phi(p) \dots} + i p^\mu W_{\widehat{A^\mu(p) \dots}} \right) \Big|_{p^2=0}.
\end{aligned} \tag{31}$$

Then one  $b$ -insertion on a physical amplitude yields

$$\begin{aligned}
\lim_{p^2=0} p^2 W_{b(p)***} &= \left( i \frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} W_{\widehat{\phi(p)***}} + i p^\mu W_{\widehat{A^\mu(p)***}} \right) \Big|_{p^2=0} \\
&= \left( -M R \Gamma_{\phi\phi} W_{\phi(p)***} + i p^\mu W_{\widehat{A^\mu(p)***}} \right) \Big|_{p^2=0} = 0,
\end{aligned} \tag{32}$$

where the \*\*\* indicates all the other *physical* states obtained via reduction formulas.

The  $\widehat{\phantom{x}}$  indicates that the external line (for instance, attached to an  $A^\mu$ ) has been removed. According to this notation

$$W_{A(p)BC \dots} = \sum_X W_{A(p)X} W_{\widehat{X(p)BC \dots}}. \tag{33}$$

### The Longitudinal Polarization

The relation with the longitudinal polarization

$$\epsilon_L = \frac{E}{M|\vec{p}|} \left( \frac{\vec{p}^2}{E}, \vec{p} \right), \quad E = \sqrt{M^2 + \vec{p}^2} \tag{34}$$

can be obtained by considering

$$\epsilon_L = \frac{E}{M|\vec{p}|} \left( |\vec{p}|, \vec{p} \right) - \frac{M}{E + |\vec{p}|} (1, \vec{0}). \quad (35)$$

It is usually assumed that

$$\epsilon_L = \frac{1}{M} \left( |\vec{p}|, \vec{p} \right) + \mathcal{O}\left(\frac{M}{E}\right) \quad (36)$$

gives the correct order of magnitude in the amplitudes

$$\epsilon_L^\mu W_{\widehat{A^\mu(p)^{***}}} \Big|_{p^2=M^2} = \frac{1}{M} p^\mu W_{\widehat{A^\mu(p)^{***}}} \Big|_{p^2=0} + \mathcal{O}\left(\frac{M}{E}\right). \quad (37)$$

Then eq. (31) reads

$$\epsilon_L^\mu W_{\widehat{A^\mu(p)^{***}}} \Big|_{p^2=M^2} = i R W_{\widehat{\phi(p)^{***}}} \Big|_{p^2=0} + \mathcal{O}\left(\frac{M}{E}\right), \quad (38)$$

which is the statement of Lee, Quigg, Thacker (1977)[2], Weldon (84)[3], Chanowitz, Gaillard (1985)[4], Gounaris, K ogerler, Neufeld (1986)[5].

Unfortunately, it will appear that the evaluation of the order of magnitude given in eq. (36) cannot always be transferred to the amplitudes as indicated by eq.(37). In particular, there is a clear evidence that the limit  $M = 0$  does not commute with the on-shell limit (reduction formula) as shown by the example with two  $b$ -insertions below.

## Two $b$ -insertions

This is a very clear example of the singular behavior of the limit  $M = 0$ . The situation is somewhat different if we use eq. (31) or (37). We use first eq. (31) and subsequently we discuss the approach by exploiting eq. (37). Note that the insertion of a second  $b_a$  line is much simpler in the Landau gauge where  $W_{A\phi} = 0$  remains valid beyond the tree approximation. In generic 't Hooft gauge there is a non-trivial mixing in the  $\phi - \partial_\mu A^\mu$  space, which causes some important technical complexities.

One has

$$\begin{aligned} \lim_{p_1^2, p_2^2=0} i^2 p_1^\mu p_2^\nu W_{\widehat{A^\mu(p_1)A^\nu(p_2)^{***}}} &= \lim_{p_1^2, p_2^2=0} p_1^2 p_2^2 \left( W_{b_1 b_2^{***}} \right. \\ &\left. + \frac{i p_1^\nu \Gamma_{\phi A^\nu}}{p_1^2} W_{\phi_1 b_2^{***}} + \frac{i p_2^\mu \Gamma_{\phi A^\mu}}{p_2^2} W_{b_1 \phi_2^{***}} + \frac{(i p_1^\nu \Gamma_{\phi A^\nu}) (i p_2^\mu \Gamma_{\phi A^\mu})}{p_1^2 p_2^2} W_{\phi_1 \phi_2^{***}} \right) \end{aligned} \quad (39)$$

The first term is zero as in eq. (29). The mixed terms can be obtained by performing the functional derivatives of the STI in eq. (13) with respect to  $\eta$  and  $K$ ,

$$W_{b_1 \phi_2^{***}} = W_{\phi_2^* \bar{c}_1^{***}}. \quad (40)$$

Thus we get (with the use of  $W_{\phi^* \bar{c}_1} = W_{b\phi}$ )

$$\begin{aligned}
& \lim_{p_1^2, p_2^2=0} i^2 p_1^\mu p_2^\nu W_{\widehat{A^\mu(p_1)} \widehat{A^\nu(p_2)}^{***}} = M^2 R^2 \lim_{p_1^2, p_2^2=0} \left( \frac{\Gamma_{\phi_1 \phi_1}}{p_1^2} p_2^2 W_{\bar{c}_2 c_2} W_{\widehat{\bar{c}_1} \widehat{c_2}^{***}} \right. \\
& \quad \left. + \frac{\Gamma_{\phi_2 \phi_2}}{p_2^2} p_1^2 W_{\bar{c}_1 c_1} W_{\widehat{\bar{c}_2} \widehat{c_1}^{***}} + W_{\widehat{\phi_1 \phi_2}^{***}} \right) \\
& = M^2 R^2 \lim_{p_1^2, p_2^2=0} \left( \bar{R} W_{\widehat{\bar{c}_1} \widehat{c_2}^{***}} + \bar{R} W_{\widehat{\bar{c}_2} \widehat{c_1}^{***}} + W_{\widehat{\phi_1 \phi_2}^{***}} \right), \tag{41}
\end{aligned}$$

where

$$\bar{R} = \lim_{p_1^2, p_2^2=0} \frac{\Gamma_{\phi_2 \phi_2}}{p_2^2} p_1^2 W_{\bar{c}_1 c_1}. \tag{42}$$

On the other hand, if we consider multi  $b$ -field insertions by using eq. (31), where the scalar mode is replaced by the longitudinal mode according to eq. (37),

$$R W_{\widehat{\phi(p)}^{***}} = \lim_{p^2=0} \frac{p^2}{M} W_{b(p)^{***}} - i \epsilon_L^\mu W_{\widehat{A^\mu(p)}^{***}} \Big|_{p^2=M^2} + \mathcal{O}\left(\frac{M}{E}\right), \tag{43}$$

we get

$$\begin{aligned}
R^2 W_{\widehat{\phi(p_1) \phi(p_2)}^{***}} & = \lim_{p_1^2, p_2^2=0} \frac{p_1^2 p_2^2}{M^2} W_{b(p_1) b(p_2)^{***}} + i \lim_{p_1^2=0} \frac{p_1^2}{M} \epsilon_L^{\mu_2} W_{b(p_1) \widehat{A^{\mu_2}(p_2)}^{***}} \Big|_{p_2^2=M^2} \\
& + i \lim_{p_2^2=0} \frac{p_2^2}{M} \epsilon_L^{\mu_1} W_{\widehat{A^{\mu_1}(p_1)} b(p_2)^{***}} \Big|_{p_1^2=M^2} + i^2 \epsilon_L^{\mu_1} \epsilon_L^{\mu_2} W_{\widehat{A^{\mu_1}(p_1) A^{\mu_2}(p_2)}^{***}} \Big|_{p_1^2, p_2^2=M^2} + \mathcal{O}\left(\frac{M}{E}\right) \\
& = -\epsilon_L^{\mu_1} \epsilon_L^{\mu_2} W_{\widehat{A^{\mu_1}(p_1) A^{\mu_2}(p_2)}^{***}} \Big|_{p_1^2, p_2^2=M^2} + \mathcal{O}\left(\frac{M}{E}\right), \tag{44}
\end{aligned}$$

where the mixed terms and the double  $b$ -insertion are zero as required by eq. (29). By replacing the scalar mode (unphysical) with the longitudinal polarization state, the value of the  $b$ -insertions changes in a substantial way.

We can conclude that the use of the substitution in eq. (31) brings in a contradiction between the results in eqs. (41) and (44). This fact has been pointed out in Ref. [4].

### Three $b$ -insertions

We consider three  $b$ -insertions, which can be relevant in processes like  $V + V \rightarrow l^+ + l^- + V$ . We use once again the eq. (31) as in eq. (39):

$$\begin{aligned}
& \lim_{p_1^2, p_2^2, p_3^2=0} \frac{i^3}{M^3} p_1^\mu p_2^\nu p_3^\rho W_{\widehat{A^\mu(p_1)} \widehat{A^\nu(p_2)} \widehat{A^\rho(p_3)}^{***}} \\
& = \lim_{p_1^2, p_2^2, p_3^2=0} p_1^2 p_2^2 p_3^2 \left( \frac{1}{M^3} W_{b_1 b_2 b_3^{***}} + \frac{R \Gamma_{\phi_1 \phi_1}}{M^2 p_1^2} W_{\phi_1 b_2 b_3^{***}} \right. \\
& \quad \left. + \frac{R \Gamma_{\phi_2 \phi_2}}{M^2 p_2^2} W_{b_1 \phi_2 b_3^{***}} + \frac{R \Gamma_{\phi_3 \phi_3}}{M^2 p_3^2} W_{b_1 b_2 \phi_3^{***}} \right)
\end{aligned} \tag{45}$$



$$\begin{aligned}
& + \frac{1}{M} \frac{R\Gamma_{\phi_1\phi_1}}{p_1^2} \frac{R\Gamma_{\phi_2\phi_2}}{p_2^2} W_{\phi_1\phi_2 b_3***} + \frac{1}{M} \frac{R\Gamma_{\phi_2\phi_2}}{p_2^2} \frac{R\Gamma_{\phi_3\phi_3}}{p_3^2} W_{b_1\phi_2\phi_3***} \\
& + \frac{1}{M} \frac{R\Gamma_{\phi_3\phi_3}}{p_3^2} \frac{R\Gamma_{\phi_1\phi_1}}{p_1^2} W_{\phi_1 b_2\phi_3***} + \frac{R\Gamma_{\phi_1\phi_1}}{p_1^2} \frac{R\Gamma_{\phi_2\phi_2}}{p_2^2} \frac{R\Gamma_{\phi_3\phi_3}}{p_3^2} W_{\phi_1\phi_2\phi_3***} \Bigg)
\end{aligned} \tag{46}$$

The mixed terms in eq. (46) are evaluated by using eq. (29):

$$\begin{aligned}
& \lim_{p_1^2, p_2^2, p_3^2=0} \frac{i^3}{M^3} p_1^\mu p_2^\nu p_3^\rho W_{\widehat{A^\mu(p_1)} \widehat{A^\nu(p_2)} \widehat{A^\rho(p_3)} ***} = \lim_{p_1^2, p_2^2, p_3^2=0} p_1^2 p_2^2 p_3^2 \\
& \left( \frac{1}{M} \frac{R\Gamma_{\phi\phi}}{p_1^2} \frac{R\Gamma_{\phi\phi}}{p_2^2} W_{\phi_1\phi_2 b_3***} + \frac{1}{M} \frac{R\Gamma_{\phi\phi}}{p_2^2} \frac{R\Gamma_{\phi\phi}}{p_3^2} W_{b_1\phi_2\phi_3***} \right. \\
& \left. + \frac{1}{M} \frac{R\Gamma_{\phi\phi}}{p_3^2} \frac{R\Gamma_{\phi\phi}}{p_1^2} W_{\phi_1 b_2\phi_3***} + \frac{R\Gamma_{\phi\phi}}{p_1^2} \frac{R\Gamma_{\phi\phi}}{p_2^2} \frac{R\Gamma_{\phi\phi}}{p_3^2} W_{\phi_1\phi_2\phi_3***} \right) \\
& = -R^3 \left( \bar{R} W_{\widehat{\phi_1} \widehat{c_2} \widehat{c_3***}} + \bar{R} W_{\widehat{c_1} \widehat{\phi_2} \widehat{c_3***}} + \bar{R} W_{\widehat{c_2} \widehat{\phi_3} \widehat{c_1***}} \right. \\
& \left. + \bar{R} W_{\widehat{\phi_2} \widehat{c_3} \widehat{c_1***}} + \bar{R} W_{\widehat{c_3} \widehat{\phi_1} \widehat{c_2***}} + \bar{R} W_{\widehat{\phi_3} \widehat{c_1} \widehat{c_2***}} + W_{\widehat{\phi_1} \widehat{\phi_2} \widehat{\phi_3***}} \right)
\end{aligned} \tag{47}$$

#### Four $b$ -insertions

There is a surprising cancellation in the case of four  $b$ -insertions.

$$\begin{aligned}
& \lim_{p_1^2 \dots p_4^2=0} p_1^\mu p_2^\nu p_3^\sigma p_4^\rho W_{\widehat{A_\mu} \widehat{A_\nu} \widehat{A_\sigma} \widehat{A_\rho}} = \lim_{p_1^2 \dots p_4^2=0} p_1^2 p_2^2 p_3^2 p_4^2 \\
& \left( W_{b_1 b_2 b_3 b_4} + M R \sum_j \frac{\Gamma_{\phi_j \phi_j}}{p_j^2} W_{b_{j+1} b_{j+2} b_{j+3} \phi_j} \right. \\
& + \frac{1}{2} M^2 R^2 \sum_{i \neq j} \frac{\Gamma_{\phi_k \phi_k}}{p_k^2} \frac{\Gamma_{\phi_l \phi_l}}{p_l^2} W_{b_i b_j \phi_k \phi_l} \\
& + M^3 R^3 \sum_j \frac{\Gamma_{\phi_{j+1} \phi_{j+1}}}{p_{j+1}^2} \frac{\Gamma_{\phi_{j+2} \phi_{j+2}}}{p_{j+2}^2} \frac{\Gamma_{\phi_{j+3} \phi_{j+3}}}{p_{j+3}^2} W_{b_j \phi_{j+1} \phi_{j+2} \phi_{j+3}} \Bigg) \\
& + M^4 R^4 \lim_{p_1^2 \dots p_4^2=0} W_{\widehat{\phi_1} \widehat{\phi_2} \widehat{\phi_3} \widehat{\phi_4}}
\end{aligned} \tag{48}$$

Now according to the eq. (29) we have

$$W_{b_1 b_2 b_3 b_4} = 0 \tag{49}$$

and

$$W_{b_c b_b b_a \phi_1} = W_{b_c b_b \phi_1^* \bar{c}_a}. \tag{50}$$

Further use of eq. (29) tells that

$$W_{b_c b_b b_a \phi_1} = W_{b_c b_b \phi_1^* \bar{c}_a} = 0. \tag{51}$$

We deal with the term with one  $b$ -insertion before considering the most difficult term. We have again from eq. (29)

$$W_{b_j \phi_{j+1} \phi_{j+2} \phi_{j+3}} = \sum_{k=1,2,3} W_{\phi_{j+k}^* \bar{c}_j \phi_{j+k+1} \phi_{j+k+2}} \quad (52)$$

Thus the relevant term in eq. (48) becomes

$$\begin{aligned} & \lim_{p_1^2 \dots p_4^2=0} p_1^2 p_2^2 p_3^2 p_4^2 M^3 R^3 \sum_j \frac{\Gamma_{\phi_{j+1}\phi_{j+1}}}{p_{j+1}^2} \frac{\Gamma_{\phi_{j+2}\phi_{j+2}}}{p_{j+2}^2} \frac{\Gamma_{\phi_{j+3}\phi_{j+3}}}{p_{j+3}^2} W_{b_j \phi_{j+1} \phi_{j+2} \phi_{j+3}} \\ &= \lim_{p_1^2 \dots p_4^2=0} p_1^2 p_2^2 p_3^2 p_4^2 M^3 R^3 \sum_j \frac{\Gamma_{\phi_{j+1}\phi_{j+1}}}{p_{j+1}^2} \frac{\Gamma_{\phi_{j+2}\phi_{j+2}}}{p_{j+2}^2} \frac{\Gamma_{\phi_{j+3}\phi_{j+3}}}{p_{j+3}^2} \\ & \quad \sum_{k=1,2,3} W_{\phi_{j+k}^* \bar{c}_j \phi_{j+k+1} \phi_{j+k+2}} \\ &= \lim_{p_1^2 \dots p_4^2=0} p_1^2 p_2^2 p_3^2 p_4^2 M^3 R^3 \sum_j \frac{\Gamma_{\phi_{j+1}\phi_{j+1}}}{p_{j+1}^2} \frac{\Gamma_{\phi_{j+2}\phi_{j+2}}}{p_{j+2}^2} \frac{\Gamma_{\phi_{j+3}\phi_{j+3}}}{p_{j+3}^2} \\ & \quad \sum_{k=1,2,3} W_{\phi^* \bar{c}(p_{j+k})} W_{\bar{c}c}(p_j) W_{\phi\phi}(p_{j+k+1}) W_{\phi\phi}(p_{j+k+2}) W_{\widehat{\bar{c}_{j+k}} \widehat{c_j} \widehat{\phi_{j+k+1}} \widehat{\phi_{j+k+2}}} \\ &= M^4 R^4 \bar{R} \sum_j \sum_{k=1,2,3} W_{\widehat{\bar{c}_{j+k}} \widehat{c_j} \widehat{\phi_{j+k+1}} \widehat{\phi_{j+k+2}}} \end{aligned} \quad (53)$$

Now we consider the most critical term in eq. (48). By using eq. (29) we get

$$W_{b_i b_j \phi_k \phi_l} = W_{\phi_k^* \bar{c}_i b_j \phi_l} + W_{\phi_l^* \bar{c}_i b_j \phi_k} \quad (54)$$

Unfortunately, one cannot remove further the  $b$ -insertion by using eq. (29). In the on-shell limit we re-express  $b$  in terms of  $\phi$  and  $\partial^\mu A_\mu$  as in the single pole contribution of eq. (31). On-shell we have from eq. (54)

$$\begin{aligned} & \lim_{p_j^2=0} p_j^2 W_{b_i b_j \phi_k \phi_l} = \lim_{p_j^2=0} \left[ MR \left( W_{\phi_k^* \bar{c}_i \widehat{\phi_j} \phi_l} + W_{\phi_l^* \bar{c}_i \widehat{\phi_j} \phi_k} \right) \right. \\ & \quad \left. + i g^2 p_j^\mu \left( W_{\phi_k^* \bar{c}_i \widehat{A_j^\mu} \phi_l} + W_{\phi_l^* \bar{c}_i \widehat{A_j^\mu} \phi_k} \right) \right] \end{aligned} \quad (55)$$

Thus the relevant terms in eq. (48) yield

$$\begin{aligned} & \lim_{p_1^2 \dots p_4^2=0} p_1^2 p_2^2 p_3^2 p_4^2 \frac{1}{2} M^2 R^2 \sum_{i \neq j} \frac{\Gamma_{\phi_k \phi_k}}{p_k^2} \frac{\Gamma_{\phi_l \phi_l}}{p_l^2} W_{b_i b_j \phi_k \phi_l} \\ &= \lim_{p_1^2 \dots p_4^2=0} \frac{1}{2} M^2 R^2 \sum_{i \neq j} p_i^2 \left( \Gamma_{\phi_k \phi_k} \Gamma_{\phi_l \phi_l} \left[ \right. \right. \\ & \quad MR \left( W_{\phi_k^* \bar{c}_i \widehat{\phi_j} \phi_l} + W_{\phi_l^* \bar{c}_i \widehat{\phi_j} \phi_k} \right) \\ & \quad \left. \left. + i g^2 p_j^\mu \left( W_{\phi_k^* \bar{c}_i \widehat{A_j^\mu} \phi_l} + W_{\phi_l^* \bar{c}_i \widehat{A_j^\mu} \phi_k} \right) \right] \right] \end{aligned}$$

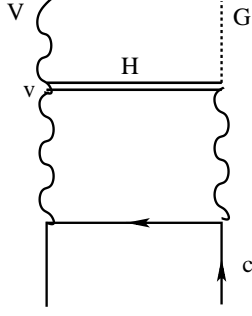


Figure 1: One-loop box diagram contributing to the second term in the r.h.s. of eq.(58)

$$\begin{aligned}
&= \lim_{p_1^2 \dots p_4^2=0} \frac{1}{2} M^2 R^2 \sum_{i \neq j} p_i^2 W_{c_i \bar{c}_i} \left[ -MR \left( \Gamma_{\phi_k \phi_k} W_{b_k \phi_k} W_{\widehat{c}_k \widehat{c}_i \widehat{\phi}_j \widehat{\phi}_l} \right. \right. \\
&\quad \left. \left. + \Gamma_{\phi_l \phi_l} W_{b_l \phi_l} W_{\widehat{c}_l \widehat{c}_i \widehat{\phi}_j \widehat{\phi}_k} \right) - ig^2 p_j^\mu \left( \Gamma_{\phi_k \phi_k} W_{b_k \phi_k} W_{\widehat{c}_k \widehat{c}_i \widehat{A}_j^\mu \widehat{\phi}_l} \right. \right. \\
&\quad \left. \left. + \Gamma_{\phi_l \phi_l} W_{b_l \phi_l} W_{\widehat{c}_l \widehat{c}_i \widehat{A}_j^\mu \widehat{\phi}_k} \right) \right] \quad (56)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{p_1^2 \dots p_4^2=0} \frac{1}{2} \bar{R} \sum_{i \neq j} \left[ -M^4 R^4 \left( W_{\widehat{c}_k \widehat{c}_i \widehat{\phi}_j \widehat{\phi}_l} + W_{\widehat{c}_l \widehat{c}_i \widehat{\phi}_j \widehat{\phi}_k} \right) \right. \\
&\quad \left. - ig^2 M^3 R^3 p_j^\mu \left( W_{\widehat{c}_k \widehat{c}_i \widehat{A}_j^\mu \widehat{\phi}_l} + W_{\widehat{c}_l \widehat{c}_i \widehat{A}_j^\mu \widehat{\phi}_k} \right) \right]. \quad (57)
\end{aligned}$$

The final result is then (eqs. (48), (49), (51), (53) and (56))

$$\begin{aligned}
&\frac{1}{M^4} \lim_{p_1^2 \dots p_4^2=0} p_1^\mu p_2^\nu p_3^\sigma p_4^\rho W_{\widehat{A}_\mu \widehat{A}_\nu \widehat{A}_\sigma \widehat{A}_\rho} = R^4 \lim_{p_1^2 \dots p_4^2=0} W_{\widehat{\phi}_1 \widehat{\phi}_2 \widehat{\phi}_3 \widehat{\phi}_4} \\
&- i \frac{g^2}{2} \lim_{p_1^2 \dots p_4^2=0} R^3 \bar{R} \sum_{i \neq j} \frac{1}{M} p_j^\mu \left( W_{\widehat{c}_k \widehat{c}_i \widehat{A}_j^\mu \widehat{\phi}_l} + W_{\widehat{c}_l \widehat{c}_i \widehat{A}_j^\mu \widehat{\phi}_k} \right). \quad (58)
\end{aligned}$$

The second term in the RHS of eq. (58) is zero in the tree approximation (this is valid in the Landau gauge, while in the 't Hooft gauge there are tree level diagrams thanks to the direct coupling of the Higgs boson and the Goldstone boson with the Faddeev-Popov ghosts). The dominant term at one loop is the box with two gauge, one Faddeev-Popov and one Higgs boson propagators shown in Figure 1. Three vertexes carry a single derivative. Then at high energy the behavior is  $p'_\mu v \mathcal{O}(\frac{1}{s})$ . Thus the total box contribution is  $\sim p^\mu p'_\mu \frac{v}{M} \mathcal{O}(\frac{1}{s})$ , i.e. of the same order as the first term on the RHS ( $\sim 1$ ).

### 3.4 Open Problems

- What is the limit theory for  $M = 0$ , if any?

- In such a limit can we use  $v$  as the order parameter?
- How does the reshuffling of the physical modes occur? In particular, does the Goldstone boson become a physical mode?
- The longitudinal mode  $\epsilon_L$  is expected to become unphysical. How?

We should give a second thought to results of Lee, Quigg, Thacker, Weldon, Chanowitz, Gaillard, Gounaris, K ogerler, Neufeld, Denner, Dittmaier, Hahn et al. [6], [7] and look if there is some clue concerning the above listed questions. Maybe lattice simulations can help in the study of the transition to  $M = 0$ . These questions might be of great phenomenological significance.

As a conclusion we would dare to say that the above mentioned very distinguished physicists have extended too much the validity of their approximations. In fact, in order to study the very high energy, they use the set of limiting Feynman rules, that are those of the massless YM theory, where the longitudinal polarization is an unphysical mode.

## 4 Part Three: Nonlinearly Realized Gauge

In this part we flash our contribution to the foundation of a quantum gauge theory, where the group of transformations is realized nonlinearly.

### 4.1 Introduction

A common structure is present in the nonlinear sigma model (NLSM), in the massive Yang-Mills (YM) model and in the Higgsless Electroweak model (EW). For  $SU(2)$  one has the action structures: NLSM action (Ref. [8]-[13])

$$S_{NLSM} = \Lambda^{D-4} \frac{M^2}{4} \int d^D x \, Tr \left\{ \partial^\mu \Omega^\dagger \partial_\mu \Omega \right\} \quad (59)$$

the St uckelberg mass for the YM model (Ref. [14]-[15])

$$S_{YM} \sim \Lambda^{D-4} M^2 \int d^D x \, Tr \left\{ \left[ A_\mu - i \Omega \partial_\mu \Omega^\dagger \right]^2 \right\} \quad (60)$$

and EW (Ref. [16]-[18]) mass terms

$$\begin{aligned} S_{EW} \sim \Lambda^{D-4} M^2 \int d^D x \left( Tr \left\{ \left( g A_\mu - \frac{g'}{2} \Omega \tau_3 B_\mu \Omega^\dagger - i \Omega \partial_\mu \Omega^\dagger \right)^2 \right\} \right. \\ \left. + \frac{\kappa}{2} \left[ Tr \left\{ g A_\mu - \frac{g'}{2} \Omega \tau_3 B_\mu \Omega^\dagger - i \Omega \partial_\mu \Omega^\dagger \tau_3 \right\} \right]^2 \right). \end{aligned} \quad (61)$$

The  $2 \times 2 \in SU(2)$  matrix may be parametrized by the real fields

$$\Omega = \phi_0 + i \tau_i \phi_i, \quad \phi_0 = \sqrt{1 - \vec{\phi}^2}. \quad (62)$$

The constraint is implemented in the path integral measure

$$\prod_x \mathcal{D}^4\phi(x) \theta(\phi_0) \delta(\vec{\phi}(x)^2 + \phi_0^2(x) - 1) = \prod_x \mathcal{D}^3\phi(x) \frac{2}{\sqrt{1 - \vec{\phi}^2}}. \quad (63)$$

The non trivial measure in the path integral is the source of very interesting facts.

The non polynomial interaction makes the theory nonrenormalizable

$$\begin{aligned} S_{NLSM} &= \Lambda^{D-4} \frac{M^2}{2} \int d^D x \left\{ \partial^\mu \phi_0 \partial_\mu \phi_0 + \partial^\mu \vec{\phi} \partial_\mu \vec{\phi} \right\} \\ &= \Lambda^{D-4} \frac{M^2}{2} \int d^D x \left\{ \partial^\mu \vec{\phi} \partial_\mu \vec{\phi} + \frac{1}{\phi_0^2} \phi_a \partial^\mu \phi_a \phi_b \partial_\mu \phi_b \right\}. \end{aligned} \quad (64)$$

Vertexes carry second power of momenta, therefore already at one loop there is an infinite number of independent divergent amplitudes. Moreover, it has been shown in the seventies and in the eighties that some divergences break chiral invariance (global) at the same order.

**Strategy:** Abandon Hamiltonian formalism and do perturbation theory directly on the effective action functional  $\Gamma$ .

## 4.2 The Local Functional Equation (LFE)

The measure is invariant under "local left multiplication" transformations  $\Omega \rightarrow U(\omega(x))\Omega$

$$\begin{aligned} \delta\phi_0 &= -\frac{\omega_a(x)}{2} \phi_a \\ \delta\phi_a &= \frac{\omega_a(x)}{2} \phi_0 + \frac{\omega_c(x)}{2} \epsilon_{abc} \phi_b. \end{aligned} \quad (65)$$

The following technical work should be done: (i) find the algebra of operators *closed* under local left multiplication transformations by starting from the classical action, (ii) associate to every composite operator an external classical source (for subtraction strategy), (iii) write the LFE which follows from the invariance of the path integral measure.

### Step (i)

This is simple in the NLSM. Introduce the "gauge field"

$$F_\mu = \frac{\tau_a}{2} F_{a\mu} \equiv i\Omega \partial_\mu \Omega^\dagger. \quad (66)$$

Its field strength tensor is zero (it describes a scalar mode) and its transformation properties are those of a gauge field:

$$F_\mu \rightarrow U F_\mu U^\dagger + iU \partial_\mu U^\dagger. \quad (67)$$

The classical action can be written as

$$S_{NLSM} = \Lambda^{D-4} \frac{M^2}{4} \int d^D x \text{Tr} \left\{ F_\mu F^\mu \right\}. \quad (68)$$

Thus the closed set of operators is  $\{\vec{\phi}, \phi_0, \vec{F}_\mu\}$ .

### Step (ii)

The complete effective action at the tree level is then

$$\Gamma^{(0)} = \Lambda^{D-4} \int d^D x \left( \frac{M^2}{8} \{F_{a\mu} - J_{a\mu}\}^2 + K_0 \phi_0 \right). \quad (69)$$

The effective action  $\Gamma[\vec{\phi}, \vec{J}_\mu, K_0]$  is obtained via the Legendre transform of the logarithm of the path integral functional

$$Z[\vec{K}, \vec{J}_\mu, K_0] \equiv \int \prod_x \frac{2}{\phi_0} \mathcal{D}^3 \phi(x) \exp \left[ \Gamma^{(0)} + \int d^D y \vec{K} \vec{\phi} \right]. \quad (70)$$

### Step (iii)

Now we exploit the invariance of the path integral measure under local left multiplication ( $\delta\phi_a = \frac{\omega_a(x)}{2}\phi_0 + \frac{\omega_c(x)}{2}\epsilon_{abc}\phi_b$ ). We expand  $\vec{\omega}(x)$  for small parameter values and obtain the LFE ( $\langle \dots \rangle$  indicates the mean over the weighted paths )

$$\begin{aligned} & \int d^D x \left\langle \left( M_D^2 (F - J)_{a\mu} (\epsilon_{abc} \omega_c F_b^\mu + \partial^\mu \omega_a) \right. \right. \\ & \left. \left. - \Lambda^{D-4} K_0 \frac{\omega_a}{2} \phi_a + \phi_0 K_a \frac{\omega_a}{2} + \epsilon_{abc} K_a \omega_c \phi_b \right) (x) \right\rangle = 0, \end{aligned} \quad (71)$$

where

$$M_D^2 \equiv \Lambda^{D-4} M^2. \quad (72)$$

We will use the notation

$$\mathcal{D}[X]_{ab}^\mu = \delta_{ab} \partial_\mu - \epsilon_{abc} X_{c\mu}. \quad (73)$$

Thus for the effective action we get the local functional equation (LFE)

$$-\partial^\mu \frac{\delta \Gamma}{\delta J_a^\mu} + \epsilon_{abc} J_c^\mu \frac{\delta \Gamma}{\delta J_b^\mu} + \frac{\Lambda^{D-4}}{2} \phi_a K_0 + \frac{1}{2\Lambda^{D-4}} \frac{\delta \Gamma}{\delta K_0} \frac{\delta \Gamma}{\delta \phi_a} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta \Gamma}{\delta \phi_b} = 0. \quad (74)$$

## 4.3 Hierarchy

The Spontaneous Breakdown of Symmetry is imposed by the condition

$$\left. \frac{\delta \Gamma}{\delta K_0} \right|_{\text{field \& sources}=0} = 1. \quad (75)$$

Then the LFE naturally induces a strong hierarchy structure among the 1PI irreducible amplitudes: all amplitudes involving the  $\vec{\phi}$  fields (descendant) are known in terms of the amplitudes involving only the (ancestor) sources  $\vec{J}_\mu$  and  $K_0$ . For instance, if we differentiate the LFE with respect to  $J_{a'}^\nu(y)$ , we get

$$\frac{M_D^2}{2} \partial^\mu \frac{\delta^2 \Gamma}{\delta J_a^\mu(x) \delta J_{a'}^\nu(y)} + \frac{\delta^2 \Gamma}{\delta \phi_a(x) \delta J_{a'}^\nu(y)} + 2\delta_{aa'} \partial_{x^\nu} \delta(x-y) = 0. \quad (76)$$

#### 4.4 Weak Power Counting (WPC)

How many ancestor divergent amplitudes are there ? The degree of divergence of a graph  $G$  for an ancestor amplitude is ( $n_L$  is the number of loops)

$$\begin{aligned}\delta(G) &= D n_L - 2I + \sum_{j,k} j V_{jk} + N_F \\ n_L &= I - \sum_{j,k} V_{jk} - N_F - N_{K_0} + 1\end{aligned}\tag{77}$$

where  $I$  is the number of propagators,  $N_F$  the number of external  $F_\mu$  sources and  $N_{K_0}$  those of  $K_0$ ;  $V_{jk}$  denotes the number of vertexes with  $k$   $\phi$ -lines and  $j$  derivatives. The superficial degree of divergence  $\delta(G)$  for a graph can be bounded by using standard arguments.

By removing  $I$  from these two equations one gets

$$\delta(G) = D n_L - 2n_L - \sum_{j,k} (2-j) V_{jk} - N_F - 2N_{K_0} + 2.\tag{78}$$

The classical action has vertexes with  $j \leq 2$ , therefore, it can be stated that

$$\delta(G) \leq n_L(D-2) + 2 - N_F - 2N_{K_0}.\tag{79}$$

For instance, at  $n_L = 1$  the only ancestor divergent (independent) amplitudes are  $(J-J)$ ,  $(J-J-J)$ ,  $(J-J-J-J)$ ,  $(K_0-J-J)$  and  $(K_0-K_0)$ . The one-loop divergences of graphs where the descendant field appears ( $\vec{\phi}$ ) are all expressible all in terms of the ancestor divergences.

#### 4.5 Perturbative Expansion

This is an *Ansatz*. Consider the generic dimension  $D$ . Start with  $\Gamma^{(0)}$ , read from it the value of the vertexes and construct  $\Gamma^{(n)}$  for  $n > 0$ . The connected amplitudes  $W^{(n)}$  can then be obtained. Few questions are in order:

1. Does  $\Gamma^{(0)}$  obey the LFE? Yes, by construction
2. Does  $\Gamma^{(n)}$ ,  $n > 0$  obey the linearized LFE ?

$$\begin{aligned}&\left( -\partial^\mu \frac{\delta}{\delta J_a^\mu} + \epsilon_{abc} J_c^\mu \frac{\delta}{\delta J_b^\mu} + \frac{1}{2\Lambda^{D-4}} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta K_0} \right. \\ &\left. + \frac{1}{2} \phi_0 \frac{\delta}{\delta \phi_a} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta}{\delta \phi_b} \right) \Gamma^{(n)} + \sum_{j=1}^{n-1} \frac{1}{2\Lambda^{D-4}} \frac{\delta \Gamma^{(j)}}{\delta \phi_a} \frac{\delta \Gamma^{(n-j)}}{\delta K_0} = 0.\end{aligned}\tag{80}$$

3. Assume that a *symmetric* subtraction procedure is given for the divergences in the limit  $D = 4$ . How does the breaking of the above equation occur?

The answers to these questions are given in a compact form by the Quantum Action Principle

$$\begin{aligned}
& \left( -\partial^\mu \frac{\delta}{\delta J_a^\mu} + \epsilon_{abc} J_c^\mu \frac{\delta}{\delta J_b^\mu} \right. \\
& \left. - \frac{\Lambda^{D-4}}{2} K_0 \frac{\delta}{\delta K_a} + \frac{1}{2\Lambda^{D-4}} K_a \frac{\delta}{\delta K_0} + \epsilon_{acb} K_c \frac{\delta}{\delta K_b} \right) Z \\
& = i \int \prod_x \frac{2}{\phi_0} \mathcal{D}^3 \phi(x) \left[ -\partial^\mu \frac{\delta \hat{\Gamma}}{\delta J_a^\mu} + \epsilon_{abc} J_c^\mu \frac{\delta \hat{\Gamma}}{\delta J_b^\mu} \right. \\
& \left. + \frac{\Lambda^{D-4}}{2} \phi_a K_0 + \frac{1}{2\Lambda^{D-4}} \frac{\delta \hat{\Gamma}}{\delta K_0} \frac{\delta \hat{\Gamma}}{\delta \phi_a} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta \hat{\Gamma}}{\delta \phi_b} \right] \exp i \left[ \hat{\Gamma} + \int d^D y \vec{K} \vec{\phi} \right], \quad (81)
\end{aligned}$$

where  $\hat{\Gamma}$  contains the counterterms  $\hat{\Gamma}^{(j)}$ ,

$$\hat{\Gamma} = \Gamma^{(0)} + \sum_{j=1}^{\infty} \hat{\Gamma}^{(j)}. \quad (82)$$

## 4.6 Subtraction Strategy

Thus if the counterterms at order  $n$  are missing, the linearized LFE is broken by the term

$$\begin{aligned}
& \left( -\partial^\mu \frac{\delta}{\delta J_a^\mu} + \epsilon_{abc} J_c^\mu \frac{\delta}{\delta J_b^\mu} + \frac{1}{2\Lambda^{D-4}} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta K_0} \right. \\
& \left. + \frac{1}{2} \phi_0 \frac{\delta}{\delta \phi_a} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta}{\delta \phi_b} \right) \Gamma^{(n)} = -\frac{1}{2\Lambda^{D-4}} \sum_{j=1}^{n-1} \frac{\delta \hat{\Gamma}^{(j)}}{\delta K_0} \frac{\delta \hat{\Gamma}^{(n-j)}}{\delta \phi_a}. \quad (83)
\end{aligned}$$

Notice that  $\frac{1}{\Lambda^{D-4}} \frac{\delta \Gamma^{(0)}}{\delta \phi_a}$  is independent from  $\Lambda^{D-4}$ . Thus we use the Laurent expansion on

$$\Lambda^{-D+4} \Gamma^{(n)} \quad (84)$$

to define the finite part and the counterterm  $\Lambda^{-D+4} \hat{\Gamma}^{(n)} = -\Lambda^{-D+4} \Gamma^{(n)} \Big|_{\text{poles}}$ .

The LFE is a power organizer of the divergences that WPC has classified. The full control can be obtained by finding the relevant local solutions of the linearized LFE

$$\begin{aligned}
& \left( -\partial^\mu \frac{\delta}{\delta J_a^\mu} + \epsilon_{abc} J_c^\mu \frac{\delta}{\delta J_b^\mu} + \frac{1}{2\Lambda^{D-4}} \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \frac{\delta}{\delta K_0} \right. \\
& \left. + \frac{1}{2} \phi_0 \frac{\delta}{\delta \phi_a} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta}{\delta \phi_b} \right) \Gamma^{(n)}[\vec{\phi}, \vec{J}_\mu, K_0] = 0. \quad (85)
\end{aligned}$$

This can easily be achieved by using the technique of *bleaching*. We shortly describe this procedure. The above equation naturally suggests the following infinitesimal transformations:

$$\delta_0 J_b^\mu = (\partial^\mu \delta_{ab} + \epsilon_{abc} J_c^\mu) \omega_a = \mathcal{D}[J]_{ba}^\mu \omega_a$$



$$\begin{aligned}
\delta_0 F_a^\mu &= \mathcal{D}[F]_{ab}^\mu \omega_b \\
\delta_0 K_0 &= -\frac{\omega_a}{\Lambda^{D-4}} \frac{\delta\Gamma^{(0)}}{\delta\phi_a} \\
\delta_0 \left( -\frac{\delta\Gamma^{(0)}}{\delta\phi_a} \right) &= \Lambda^{D-4} \frac{1}{2} \omega_a K_0 + \frac{1}{2} \epsilon_{abc} \omega_c \left( -\frac{\delta\Gamma^{(0)}}{\delta\phi_b} \right),
\end{aligned} \tag{86}$$

which lead to the bleaching

$$\begin{aligned}
\mathfrak{J}_\mu &\equiv \Omega^\dagger (J_\mu - F_\mu) \Omega \\
\mathfrak{K}_0 &\equiv \frac{K_0}{\phi_0} - \frac{M^2}{4} (J_b^\mu - F_b^\mu) \frac{\partial F_{b\mu}}{\partial \phi_a} \phi_a
\end{aligned} \tag{87}$$

Here are few facts about bleaching. i) The relations are invertible, ii) In the case of  $\mathfrak{J}_{a\mu}$ , bleaching is a kind of gauge transformation where the parameters are the  $\vec{\phi}$  fields:

$$\begin{aligned}
\mathfrak{J}_\mu &= \Omega^\dagger J_\mu \Omega + i\Omega \partial_\mu \Omega \\
\partial_\mu \mathfrak{J}_\nu &= \Omega^\dagger \left( \partial_\mu + \Omega \partial_\mu \Omega^\dagger \right) (J_\nu - F_\nu) \Omega = \Omega^\dagger \mathcal{D}_\mu[F] (J_\nu - F_\nu) \Omega
\end{aligned} \tag{88}$$

iii) the invariants can be constructed by using  $\mathfrak{J}_\mu$  and  $\mathfrak{K}_0$  and their space-time derivatives.

Ancestor amplitudes do not depend explicitly on  $\vec{\phi}$ . We consider only those relevant for the one-loop divergences.

We give here a list of the relevant one-loop invariants necessary for the parameterization of the one-loop divergences of the NLSM:

$$\begin{aligned}
\mathcal{I}_1 &= \int d^D x \left[ D_\mu (F - J)_\nu \right]_a \left[ D^\mu (F - J)^\nu \right]_a, \\
\mathcal{I}_2 &= \int d^D x \left[ D_\mu (F - J)^\mu \right]_a \left[ D_\nu (F - J)^\nu \right]_a, \\
\mathcal{I}_3 &= \int d^D x \epsilon_{abc} \left[ D_\mu (F - J)_\nu \right]_a \left( F_b^\mu - J_b^\mu \right) \left( F_c^\nu - J_c^\nu \right), \\
\mathcal{I}_4 &= \int d^D x \left( \frac{K_0}{\phi_0} + \frac{M^2}{4} [F_b^\mu - J_b^\mu] \frac{\partial F_{b\mu}}{\partial \phi_a} \phi_a \right)^2, \\
\mathcal{I}_5 &= \int d^D x \left( \frac{K_0}{\phi_0} + \frac{M^2}{4} [F_b^\mu - J_b^\mu] \frac{\partial F_{b\mu}}{\partial \phi_a} \phi_a \right) \left( F_c^\mu - J_c^\mu \right)^2, \\
\mathcal{I}_6 &= \int d^D x \left( F_a^\mu - J_a^\mu \right)^2 \left( F_b^\nu - J_b^\nu \right)^2, \\
\mathcal{I}_7 &= \int d^D x \left( F_a^\mu - J_a^\mu \right) \left( F_a^\nu - J_a^\nu \right) \left( F_{b\mu} - J_{b\mu} \right) \left( F_{b\nu} - J_{b\nu} \right),
\end{aligned} \tag{89}$$

where  $D_\mu$  denotes the covariant derivative w.r.t  $F_{a\mu}$ :

$$D_{ab\mu} = \delta_{ab} \partial_\mu - \epsilon_{abc} F_{c\mu}. \tag{90}$$

The counterterms are evaluated by extracting the pole parts from the relevant amplitudes given by the effective action functional normalized by  $\Lambda^{-D+4}\Gamma$ . It is very important to care about the relation

$$2(\mathcal{I}_1 - \mathcal{I}_2) - 4\mathcal{I}_3 + (\mathcal{I}_6 - \mathcal{I}_7) = \int d^D x \mathcal{G}_{a\mu\nu}[\mathfrak{J}] \mathcal{G}_a^{\mu\nu}[\mathfrak{J}] = \int d^D x \mathcal{G}_{a\mu\nu}[J] \mathcal{G}_a^{\mu\nu}[J] \approx 0. \quad (91)$$

The last integral is sterile: no descendant terms are generated. Now the calculation gives

$$\Gamma^{(1)} = \frac{1}{D-4} \frac{\Lambda^{D-4}}{(4\pi)^2} \left[ -\frac{1}{12} (\mathcal{I}_1 - \mathcal{I}_2 - \mathcal{I}_3) + \frac{1}{48} (\mathcal{I}_6 + 2\mathcal{I}_7) + \frac{3}{2} \frac{1}{M^4} \mathcal{I}_4 + \frac{1}{2} \frac{1}{M^2} \mathcal{I}_5 \right]. \quad (92)$$

## 4.7 The massive Yang-Mills theory

$\Omega$  describes the Goldstone bosons, that are here unphysical modes. Then it is important to ensure that the Slavnov-Taylor Identity (STI) is valid in order to preserve unitarity. The LFE must be compatible with the STI. A suitable gauge-fixing term will help to achieve this result. The Landau gauge is the simplest, since the tadpole contributions can be neglected in most cases. The transformations to be considered are the local left  $SU(2)_L$  and the global right  $SU(2)_R$  on  $\Omega$ , the gauge fields are  $A_\mu$  and the Faddeev-Popov fields are  $c, \bar{c}$ . Few external sources are needed in order to describe the complete (under the  $SU(2)_L \otimes SU(2)_R$ ) set of composite operators.

The action in the presence of the Landau gauge-fixing terms looks as follows:

$$\begin{aligned} \Gamma^{(0)} &= S_{YM} + \frac{\Lambda^{D-4}}{g^2} \int d^D x \left( B_a(D^\mu[V](A_\mu - V_\mu))_a - \bar{c}_a(D^\mu[V]D_\mu[A]c)_a \right) \\ &\quad + \int d^D x \left( A_{a\mu}^* s A_a^\mu + \phi_0^* s \phi_0 + \phi_a^* s \phi_a + c_a^* s c_a + K_0 \phi_0 \right). \end{aligned} \quad (93)$$

$$S_{YM} = \frac{\Lambda^{(D-4)}}{g^2} \int d^D x \left( -\frac{1}{4} G_{a\mu\nu}[A] G_a^{\mu\nu}[A] + \frac{M^2}{2} (A_{a\mu} - F_{a\mu})^2 \right). \quad (94)$$

$$\Omega = \frac{1}{v} (\phi_0 + i\tau_a \phi_a), \quad \phi_0^2 + \phi_a^2 = v^2 \quad (95)$$

where  $v$  is a parameter with dimension one. We stress that  $v$  is *not a parameter of the model*, because it can be removed by a rescaling of the fields  $\vec{\phi}$  and  $\phi_0$ .

## Slavnov-Taylor Identity

The  $S$ -matrix satisfies the following equation at the *perturbative level*:

$$\langle \alpha | \beta \rangle = \sum_{n \in \{\text{physical states}\}} \langle \alpha | S | n \rangle \langle n | S^\dagger | \beta \rangle$$

if both  $\alpha$  and  $\beta$  are physical states. This in general is valid if the Slavnov-Taylor identity is valid.

$$\mathcal{S}(\Gamma) = \int d^D x \left( \frac{\delta\Gamma}{\delta A_{a\mu}^*} \frac{\delta\Gamma}{\delta A_a^\mu} + \frac{\delta\Gamma}{\delta \phi_a^*} \frac{\delta\Gamma}{\delta \phi_a} + \frac{\delta\Gamma}{\delta c_a^*} \frac{\delta\Gamma}{\delta c_a} + B_a \frac{\delta\Gamma}{\delta \bar{c}_a} - K_0 \frac{\delta\Gamma}{\delta \phi_0^*} \right) = 0. \quad (96)$$

The LFE for the massive YM model can be cast in the form:

$$\begin{aligned} \mathcal{W}(\Gamma) \equiv \int d^D x \alpha_a^L(x) & \left( -\partial_\mu \frac{\delta\Gamma}{\delta V_{a\mu}} + \epsilon_{abc} V_{c\mu} \frac{\delta\Gamma}{\delta V_{b\mu}} - \partial_\mu \frac{\delta\Gamma}{\delta A_{a\mu}} \right. \\ & + \epsilon_{abc} A_{c\mu} \frac{\delta\Gamma}{\delta A_{b\mu}} + \epsilon_{abc} B_c \frac{\delta\Gamma}{\delta B_b} + \frac{1}{2} K_0 \phi_a + \frac{1}{2} \frac{\delta\Gamma}{\delta K_0} \frac{\delta\Gamma}{\delta \phi_a} \\ & + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta\Gamma}{\delta \phi_b} + \epsilon_{abc} \bar{c}_c \frac{\delta\Gamma}{\delta \bar{c}_b} + \epsilon_{abc} c_c \frac{\delta\Gamma}{\delta c_b} \\ & + \epsilon_{abc} A_{c\mu}^* \frac{\delta\Gamma}{\delta A_{b\mu}^*} + \epsilon_{abc} c_c^* \frac{\delta\Gamma}{\delta c_b^*} + \frac{1}{2} \phi_0^* \frac{\delta\Gamma}{\delta \phi_a^*} \\ & \left. + \frac{1}{2} \epsilon_{abc} \phi_c^* \frac{\delta\Gamma}{\delta \phi_b^*} - \frac{1}{2} \phi_a^* \frac{\delta\Gamma}{\delta \phi_0^*} \right) = 0. \end{aligned} \quad (97)$$

$\Gamma$  also obeys the Landau gauge equation

$$\frac{\delta\Gamma}{\delta B_a} = \frac{\Lambda^{D-4}}{g^2} D^\mu[V](A_\mu - V_\mu)_a \quad (98)$$

## Linearized Equations and Induced Transformations

The structure of both STI and LFE is standard. Thus we can

1. Establish the full hierarchy (only the Goldstone bosons are descendant fields)
2. Confirm the validity of the WPC
3. Introduce the linearized STI and LFE
4. Extract from the linearized STI and LFE the generators of the transformations on the effective action  $\Gamma$
5. Check that the generators stemming from STI commute with those from LFE

## Subtraction procedure

With these tools we can construct the most general classical action compatible with the WPC and the invariance under the BRST transformations and the LFE induced symmetry. Surprisingly enough, the resulting action is the standard YM field theory with a Stückelberg mass term.

The subtraction procedure of the divergences is then the same as in the NLSM: subtraction of the pure pole parts in the Laurent expansion around  $D = 4$  of the normalized amplitudes  $\Lambda^{-D+4}\Gamma$ . This subtraction procedure has been implemented in the one-loop calculation of the gauge field two-point functions [15], [18]. Moreover, it has been tested for a solvable model [19].

## Consistency of the Subtraction Procedure

The two-loop self-energy amplitude has been considered from the point of view of the consistency. It has been argued that the subtraction scheme is consistent: i) the counterterms are local ii) physical unitarity is satisfied iii) the STI and LFE induced symmetry on  $\Gamma$  is preserved.

In Ref. [15] we proved the following results:

- 1) explicit calculation of the gauge field two-point function.
- 2) Check that the counterterms are local at the two-loop level.
- 3) Validity of unitarity.
- 4) All divergences (infinite) at the one-loop level are subtracted by a finite number of counterterms.

## Outlook and (some) open questions

Several issues should be addressed:

- Phenomenological applications
- Running constant (dependence on  $\Lambda$ )
- How to proceed with a generic regularization tool?
- Well-defined strategy of minimal subtraction with anticommuting  $\gamma_5$ .
- Extension to Grand Unified groups

## 5 Conclusions

Our approach to theories with nonlinearly realized gauge group is based on the Local Functional Equation, which applies to the generating functionals. The features of this method are quite novel in field theory and can be briefly summarized as follows:

- Hierarchy: all the amplitudes involving the parameter fields (the pion field in the nonlinear sigma model, the Goldstone bosons in the nonabelian gauge theories) can be derived from well-defined ancestor field amplitudes given in terms of gauge- and order-parameter-fields. This property allows one to fix at every order an infinite number of divergent amplitudes in terms of a finite number of divergences involving only the ancestor fields.
- Weak Power Counting: for the ancestor amplitudes a criterion is needed in order to make hierarchy effective. The subtraction procedure that we are implementing is compatible with the WPC, i.e. if the starting action is constructed by using the WPC, then the counterterms do not alter this property.

- Existence of a consistent subtraction procedure (symmetric and local): it can be proven that minimal dimensional subtraction on properly normalized amplitudes maintains the validity of the LFE.
- Necessity of a finite number of physical parameters. It is essential that the number of free parameters is finite and independent from the order in the loop expansion. Otherwise the subtraction strategy would not be consistent, since every parameter should be present in the tree-level action.

For massive Yang-Mills theory, using Slavnov-Taylor identities and the Landau gauge equation we proved

- The physical unitarity of the theory. This property is of paramount importance since our approach, as in the usual linear case, has unphysical modes (Goldstone bosons, spin-zero vector field polarization, Faddeev-Popov ghosts). The proof proceeds in the standard way by showing that the unphysical modes cancel in the unitarity equation for the S-matrix involving only physical states.
- The consistency of the Local Functional Equation with all other equations, such as the Slavnov-Taylor identities, the gauge-fixing equation and the anti-ghost equation. All the equations are not spoiled in the proposed subtraction procedure.
- We finally mention that the massive YM theory can also be formulated in the 't Hooft-Feynman gauge. However, in this gauge one has to deal with many tadpole diagrams that are absent in the Landau gauge.

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